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Processes of Problem Solving

Problem solving is an integral part of all mathematics learning. To solve problems, students must draw upon their knowledge of the concepts and skills they have learned and apply them to a novel situation; through this process, students develop new mathematical understanding. Problem solving should not be an isolated part of the program. Rather, problem solving should involve all content areas, numbers and operations, algebra, geometry, measurement, and data analysis and probability.

Instructional programs from kindergarten through grade 12 should enable students to:

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and other contexts
- Apply and adapt a variety of appropriate strategies to solve problems, and
- Monitor and reflect on the process of problem solving.

To build mathematical knowledge through problem solving, teachers must first be able to choose good problems that give students the chance to solidify and extend what they know and that stimulate mathematics learning. In selecting worthwhile problems that help develop important mathematical ideas, teachers need to understand how to analyze and adapt problems, be able to focus on mathematical ideas that can be illuminated by working on the problem, and learn to anticipate students' questions to decide if particular problems will further develop the mathematical goals for the class.

Good problem solvers tend naturally to analyze situations carefully in mathematical terms and to properly pose problems based on particular situations. They consider simple cases before trying something more complicated, and they look for and examine patterns and relationships. They develop a disposition to analyze more deeply that leads to a more complete understanding of the situation and the correct solution. Throughout the grades, teachers build this disposition by asking questions that help students find the mathematics in their experiences, and by encouraging students to persist with interesting but challenging problems.

Students who can successfully solve problems are able to apply and adapt a variety of appropriate strategies. These strategies must receive instructional attention to assist students to learn them, and opportunities to use these strategies must be embedded naturally in the curriculum across the content areas. In the earliest grades, students first learn to express, categorize, and compare their strategies. In the middle grades, students should be skilled in recognizing when various strategies are appropriate to use and should be capable of deciding how to use them. By high school, students should have access to a wide range of strategies, be able to decide which one to use, and be able to adapt and invent strategies.

Effective problem solvers constantly monitor and adjust what they are doing. They make sure they understand the problem and they plan frequently, periodically taking stock of their progress to see whether they seem to be on the right track. If they are not making progress, they stop to consider alternatives and do not hesitate to take a completely different approach. Poor problem solving is often due not to lack of knowledge but to the ineffective use of what they do know. Good problem solvers become aware of what they are doing and frequently use reflective skills. Teachers who create classroom environments that support the development of reflective habits provide opportunities where students are more likely to monitor their understanding and more likely to make productive adjustments necessary when solving problems.

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Reasoning and Proof

Reasoning is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas at all grade levels, students will learn that mathematics makes sense. Reasoning and proof cannot simply be taught in a single unit on logic or by ‘doing proofs’ in geometry. Reasoning and proof should be a consistent part of students’ mathematical experience in kindergarten through grade 12. Reasoning mathematically is a habit of mind, and is developed through consistent use in many contexts.

Systematic reasoning is a defining feature of mathematics. It is found in all content areas and, with different degrees of rigor, at all grade levels. Effective instructional programs should enable students to:

- Recognize reasoning and proof as fundamental aspects of mathematics,
- Make and investigate mathematical conjectures,
- Develop and evaluate mathematical arguments and proofs, and
- Select and use various types of reasoning and methods of proof.

Conjecture, which is informed guessing, is a major pathway to discovery. Students can learn to make, refine, and test conjectures in very early grades, and they can develop their abilities to investigate their conjectures using concrete materials, calculators and other tools, and increasingly through the grades, mathematical representations and symbols. Beginning in the elementary grades, students can learn to disprove conjectures by finding counterexamples. At all levels, students will reason inductively from specific cases to larger patterns. Increasingly, they should learn to make effective deductive arguments. Students also need to work with other students to formulate their conjectures and to listen to and understand conjectures and explanations offered by classmates.

Early elementary students tend to justify general claims using specific cases. By the upper elementary grades, justifications can become more generalized and can draw on other mathematical results. In high school, students should be expected to construct relatively complex chains of reasoning. As students move through the grades they compare their ideas with other’s ideas, which may cause them to modify, consolidate, or strengthen their arguments or reasoning.

Students need to learn how to select and use various types of reasoning and methods of proof to solve problems in a wide array of contexts. Students begin with informal reasoning, compared to formal logical deduction, and gradually become adept at various types of reasoning – algebraic and geometric reasoning, proportional reasoning, probabilistic reasoning, statistical reasoning, and so forth. Students need to encounter and build proficiency in all these areas with increasing sophistication as they move through the curriculum.

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Communication

Though communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process helps build meaning and permanence for ideas and makes them public. When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing. Listening to others' explanations gives students opportunities to develop their own understanding. Conversations in which mathematical ideas are explored from multiple perspectives help students sharpen their thinking and make connections. Such activity also helps students to develop a language for expressing mathematical ideas and an appreciation for the need for precision in that language.

Instructional programs from kindergarten through grade 12 should enable students to:

- Organize and consolidate their thinking through communication,
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others,
- Analyze and evaluate the mathematical thinking and strategies of others,
- Use the language of mathematics to express mathematical ideas precisely, and
- Describe mathematical concepts using developmentally appropriate definitions.

Reflection and communication are intertwined processes in mathematical learning. Writing in mathematics can also help students consolidate their thinking because it requires them to reflect on their work and clarify their thoughts about the ideas developed. Examining and discussing both exemplary and problematic pieces of mathematical writing can be beneficial at all levels. Students also need to test their ideas on the basis of shared knowledge and see whether they can be understood and are convincing.

Learning what is acceptable as evidence in mathematics should be an instructional goal from kindergarten through grade 12. Students benefit from analyzing and evaluating the mathematical thinking and strategies of others. Since not all methods and ideas have equal merit, students must learn to examine the methods and ideas of others to determine their own strengths and limitations.

As students articulate their mathematical understanding, they begin by using everyday, familiar language. Building upon this base, teachers can help students see that some words are also used in mathematics with different or more-precise meanings. While it is important that students have experiences that help them appreciate the power and precision of mathematical language, it is important to avoid imposing formal mathematical language prematurely. Students must first be allowed to grapple with their ideas and develop their own informal means of expressing them. Technology affords other opportunities and challenges for the development and analysis of language. Spreadsheets, algebraic symbols, and geometric shapes all contribute to building the vocabulary and an understanding of the language of mathematics. Graphing calculators, spreadsheets, dynamic software, computer algebra systems and various web-based applications can contribute to mathematical understanding and communication.

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Connections

Understanding involves making connections. Students that can connect their mathematical ideas develop a deeper and more lasting understanding of the rich interplay among mathematical topics and about the utility of mathematics. Viewing mathematics as a whole highlights the need for studying and thinking about the connections within the discipline, as reflected within the grade-level curriculum and across grades. Teachers help students build a disposition to recognize and use connections among mathematical ideas by asking guiding questions and providing opportunities for students to integrate mathematics in many contexts. Students begin to see the connections between arithmetic operations, understanding, for example, how multiplication can be thought of as repeated addition. As they see how mathematical operations can be used in different contexts, they develop an appreciation for the abstraction of mathematics.

As students progress through their school mathematics experience, their ability to see the same mathematical structure in different settings should increase. Students in Kindergarten through grade 2 recognize instances of counting, number, and shape. Upper elementary school students look for instances of arithmetic operations, and middle-grade students look for examples of rational numbers, proportionality, and linear relationships. High school students look for connections among the many mathematical ideas they are encountering.

As students develop a view of mathematics as a connected and integrated whole, they will have less of a tendency to view mathematical skills and concepts separately. When conceptual understanding is linked to procedures, students will not perceive mathematics as an arbitrary set of rules. This integration of procedures and concepts should be central in school mathematics.

Instructional programs from kindergarten through grade 12 should enable students to:

- Recognize and use connections among mathematical ideas,
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole, and
- Recognize and apply mathematics in contexts outside of mathematics.

The opportunity for students to experience mathematics in a context is important. Mathematics is used in science, the social sciences, medicine, engineering, construction, business, government, arts and architecture, finance and commerce, and many other fields. These links are not only through content, but also, as in the case of science, through process. The processes and content of science can inspire an approach to solving problems that applies to mathematics, or even result in the creation of new mathematical fields. Equally, students who see the connection of mathematics to the world and to other disciplines are better able to apply knowledge from several different areas and are more likely to be successful problem solvers.

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Representation

The term representation refers both to process and product, meaning the act of capturing a mathematical concept or relationship in some form or the form itself. Representing applies to externally observable processes and products as well as to those that occur ‘internally’ in the minds of students as they are doing mathematics. When students have access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically.

Although some forms of representation, such as diagrams, graphical displays, and symbolic expressions, have long been a part of school mathematics, they have often been taught and learned in isolation. Representations should be treated as essential elements in supporting students’ understanding of mathematical concepts and relationships, in communicating mathematical approaches, arguments, in understanding one’s self and others, in recognizing connections among related mathematical concepts, and in applying mathematics to realistic problem situations through modeling. It is important to encourage students to represent their ideas in ways that makes sense to them, first in ways that are not conventional and later, in conventional form.

Instructional programs from kindergarten through grade 12 should enable all students to:

- Create and use representations to organize, record, and communicate mathematical ideas,
- Select, apply, and translate among mathematical representations to solve problems, and
- Use representations to model and interpret physical, social, and mathematical phenomena.

Teachers gain valuable insight into students’ ways of interpreting and thinking about mathematics by looking at their representations. Teachers can then build bridges from students’ personal representations to more conventional ones when appropriate. It is important that students have opportunities to not only learn conventional forms of representation, but also to construct, refine, and use their own representations as tools to support learning and doing mathematics.

Computers and calculators change what students can do with conventional representations and expand the set of representations with which they can work. A variety of technological tools allow students to manipulate, visualize, and simulate more complex data and therefore represent and investigate mathematical ideas and situations not otherwise possible. As students’ representational repertoire expands, it is important for students to reflect on their use of representations to develop an understanding of the relative strengths and weaknesses of various representations for various purposes.

Different representations can illuminate different aspects of a complex concept or relationship. To become deeply knowledgeable about many aspects of mathematics, students need a variety of representations to support their understanding. As they move through grades, students’ repertoires of representations should expand to include more complex pictures, tables, graphs, and words to model problems and situations. As students become more mathematically sophisticated, they develop an increasingly large array of mathematical representations as well as the knowledge of how to use them productively.

The term model has many meanings. It can refer to physical materials, that is, manipulative models. It also refers to providing example behavior, such as when a teacher demonstrates a problem-solving process, or model can be used synonymously with representation. The term mathematical model means a mathematical representation of elements and relationships in an idealized version of a complex phenomenon. Mathematical models

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can be used to clarify and interpret the phenomenon and to solve problems.

In the early grades, students model situations using physical objects and simple pictures. As middle grade students model and solve problems that arise in the real and the mathematical worlds, they learn to use variables to represent unknowns and also learn how to employ equations, graphs, and tables to represent and analyze situations. High school students create and interpret models of phenomena drawn from a wider range of contexts by identifying the essential elements of the context and by devising representations that capture mathematical relationships among those elements. With technology tools, students can explore and understand complex concepts. These tools now allow students to explore iterative models for situations that were once studied in much more advanced courses.

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